

A new approach toward locally bounded global solutions to a $3D$ chemotaxis-stokes system with nonlinear diffusion and rotation

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Abstract

We consider a degenerate quasilinear chemotaxis–Stokes type involving rotation in the aggregative term,

$$\begin{cases} n_t + u \cdot \nabla n = \Delta n^m - \nabla \cdot (nS(x, n, c) \cdot \nabla c), & x \in \Omega, t > 0, \\ c_t + u \cdot \nabla c = \Delta c - nc, & x \in \Omega, t > 0, \\ u_t + \nabla P = \Delta u + n \nabla \phi, & x \in \Omega, t > 0, \\ \nabla \cdot u = 0, & x \in \Omega, t > 0, \end{cases} \quad (CF)$$

where $\Omega \subseteq \mathbb{R}^3$ is a bounded convex domain with smooth boundary. Here $S \in C^2(\bar{\Omega} \times [0, \infty)^2; \mathbb{R}^{3 \times 3})$ is a matrix with $s_{i,j} \in C^1(\bar{\Omega} \times [0, \infty) \times [0, \infty))$. Moreover, $|S(x, n, c)| \leq S_0(c)$ for all $(x, n, c) \in \bar{\Omega} \times [0, \infty) \times [0, \infty)$ with $S_0(c)$ nondecreasing on $[0, \infty)$. If

$$m > \frac{9}{8},$$

then for all reasonably regular initial data, a corresponding initial-boundary value problem for (CF) possesses a globally defined weak solution (n, c, u) . Moreover, for any fixed $T > 0$ this solution is bounded in $\Omega \times (0, T)$ in the sense that

$$\|u(\cdot, t)\|_{L^\infty(\Omega)} + \|c(\cdot, t)\|_{W^{1,\infty}(\Omega)} + \|n(\cdot, t)\|_{L^\infty(\Omega)} \leq C \quad \text{for all } t \in (0, T)$$

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is valid with some $C(T) > 0$. In particular, if $S(x, n, c) := C_S$, this result extends of Tao and Winkler ([15]), while, if fluid-free subcase of the flow of fluid is ignored or the fluid is stationary in (CF) , $S(x, n, c) := C_S$ and $N = 3$, this results is consistent with the result of Theorem 2.1 of Zheng and Wang ([28]). In view of some carefully analysis, we can establish some natural gradient-like structure of the functional $\int_{\Omega} n(\cdot, t) \ln n(\cdot, t) + \int_{\Omega} |\nabla \sqrt{c}(\cdot, t)|^2 + \int_{\Omega} |u(\cdot, t)|^2$ of (CF) , which is a new estimate of chemotaxis–Stokes system with rotation (see [1, 2, 22, 19]).

Key words: Chemotaxis–fluid system; Global existence; Tensor-valued sensitivity

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1 Introduction

We consider the following chemotaxis-Stokes system with porous medium diffusion and rotation in the aggregation term:

$$\left\{ \begin{array}{l} n_t + u \cdot \nabla n = \nabla \cdot (D(n) \nabla n) - \nabla \cdot (nS(x, n, c) \cdot \nabla c), \quad x \in \Omega, t > 0, \\ c_t + u \cdot \nabla c = \Delta c - nc, \quad x \in \Omega, t > 0, \\ u_t + \nabla P = \Delta u + n \nabla \phi, \quad x \in \Omega, t > 0, \\ \nabla \cdot u = 0, \quad x \in \Omega, t > 0, \\ (\nabla n - nS(x, n, c) \cdot \nabla c) \cdot \nu = \nabla c \cdot \nu = 0, u = 0, \quad x \in \partial\Omega, t > 0, \\ n(x, 0) = n_0(x), c(x, 0) = c_0(x), u(x, 0) = u_0(x), \quad x \in \Omega, \end{array} \right. \quad (1.1)$$

where Ω is a bounded convex domain in \mathbb{R}^3 with smooth boundary $\partial\Omega$, $S(x, n, c)$ is a chemotactic sensitivity tensor satisfying

$$S \in C^2(\bar{\Omega} \times [0, \infty)^2; \mathbb{R}^{3 \times 3}) \quad (1.2)$$

and

$$|S(x, n, c)| \leq S_0(c) \quad \text{for all } (x, n, c) \in \Omega \times [0, \infty)^2 \quad (1.3)$$

with some nondecreasing $S_0 : [0, \infty) \rightarrow \mathbb{R}$, porous medium diffusion function D satisfies

$$D \in C_{loc}^\iota([0, \infty)) \text{ for some } \iota > 0, \quad D(n) \geq C_D n^{m-1} \text{ for all } n > 0 \quad (1.4)$$

with some $m \geq 1$. Here n and c denote the bacterium density and the oxygen concentration, respectively, and u represents the velocity field of the fluid subject to an incompressible Navier-Stokes equation with pressure P and viscosity η and a gravitational force $\nabla \phi$. This type of the system arises in mathematical biology to model the evolution of oxygen-driven swimming bacteria in an incompressible fluid.

To motivate our study, let us first recall the following fluid-free subcase of system (1.1):

$$\left\{ \begin{array}{l} n_t = \nabla \cdot (D(n) \nabla n) - \nabla \cdot (nS(x, n, c) \cdot \nabla c), \quad x \in \Omega, t > 0, \\ c_t = \Delta c - nc, \quad x \in \Omega, t > 0, \\ \nabla n \cdot \nu = \nabla c \cdot \nu = 0, \quad x \in \partial\Omega, t > 0, \\ n(x, 0) = n_0(x), c(x, 0) = c_0(x), \quad x \in \Omega. \end{array} \right. \quad (1.5)$$

There are only few rigorous results on global existence and qualitative behavior of solutions to (1.5) with either a matrix-valued function ($S(x, n, c)$) or a scalar one ($S(x, n, c) := S(c)$) (see e.g. [13, 28, 23]).

The following chemotaxis-(Navier)-Stokes model which is a generalized version of the model proposed in [16], describes the motion of oxygen-driven swimming cells in an incompressible fluid:

$$\begin{cases} n_t + u \cdot \nabla n = \nabla \cdot (D(n) \nabla n) - C_S \nabla \cdot (n S(c) \cdot \nabla c), & x \in \Omega, t > 0, \\ c_t + u \cdot \nabla c = \Delta c - nc, & x \in \Omega, t > 0, \\ u_t + (u \cdot \nabla u) = \nabla P + \Delta u + n \nabla \phi, & x \in \Omega, t > 0, \\ \nabla \cdot u = 0, & x \in \Omega, t > 0, \end{cases} \quad (1.6)$$

where compared with (1.1), the nonlinear convective term $u \nabla u$ exists u -equation of (1.6), moreover $S(x, n, c) := S(c)$ is scalar function in (1.6) and hence there is a certain natural quasi-Lyapunov functionals of (1.6). Hence, by making use of energy-type functionals, some local and global solvability of corresponding initial value problem for (1.6) in either bounded or unbounded domains have been obtained in the past years (see e.g. Lorz et al. [6, 10], Winkler et al. [1, 14, 21], Chae et al. [4, 5], Di Francesco et al. ([7], Zhang, Zheng [25] and references therein).

If the chemotactic sensitivity $S(x, n, c)$ is regarded as a tensor rather than a scalar one ([24]), (1.1) turns into a chemotaxis-Stokes system with rotational flux which implies that chemotactic migration need not be directed along the gradient of signal concentration. In contrast to the chemotaxis-fluid system (1.6), chemotaxis-fluid systems with tensor-valued sensitivity lose some natural gradient-like structure (see Cao [2], Wang et al. [18, 19], Winkler [22]). This gives rise to considerable mathematical difficulties. Therefore, only very few results appear to be available on chemotaxis-Stokes system with such tensor-valued sensitivities (Cao et al. [2], Ishida [8], Wang et al. [18, 20, 19], Winkler [22]). In fact, assuming (1.2)–(1.3) holds, Ishida ([8]) showed that the corresponding full chemotaxis Navier-Stokes system with porous-medium-type diffusion model possesses a bounded global weak solution in two space dimensions. While, in three space dimensions, if the initial data satisfy certain **smallness** conditions and $D(n) = 1$, Cao and Lankeit [3] showed that (1.1) has global clas-

sical solutions and give decay properties of these solutions. In this paper, the core step is to establish the estimates of the functional

$$\int_{\Omega} n^p(\cdot, t) + \int_{\Omega} |\nabla c(\cdot, t)|^{2q} \quad (1.7)$$

for suitably chosen but arbitrarily large numbers $p > 1$ and $q > 1$. In fact, one of our main tool is consideration of the natural gradient-like energy functional

$$\int_{\Omega} n_{\varepsilon}(\cdot, t) \ln n_{\varepsilon}(\cdot, t) + \int_{\Omega} |\nabla \sqrt{c_{\varepsilon}}(\cdot, t)|^2 + \int_{\Omega} |u_{\varepsilon}(\cdot, t)|^2, \quad (1.8)$$

which is new estimate of chemotaxis–Stokes system **with rotation** (see Lemmata 2.7–2.10), although, (1.8) has been used to solve the chemotaxis–(Navier)–Stokes system **without rotation** (see [1, 9, 21]). Here $(n_{\varepsilon}, c_{\varepsilon}, u_{\varepsilon})$ is solution of the approximate problem of (1.1). We guess that (1.8) can also be dealt with other types of systems, e.g., quasilinear chemotaxis system with rotation, chemotaxis–(Navier)–Stokes system with rotation. Then, in view of the estimates (1.8), the suitable interpolation arguments (see Lemma 2.4) and the basic a priori information (see Lemma 2.5), we can get the the estimates of the functional

$$\int_{\Omega} n_{\varepsilon}^p + \int_{\Omega} |\nabla c_{\varepsilon}|^{2q_0} + \int_{\Omega} |A^{\frac{1}{2}} u_{\varepsilon}|^2, \quad (1.9)$$

where $p := p(q_0, m) \geq \frac{7}{4}$ and $q_0 < 2$. Next, (1.9) and some other carefully analysis (Lemma 2.18–2.19) yield the estimates of the functional (1.7). Indeed, the paper makes sure that under the assumption m satisfies (1.4), S satisfies (1.2) and (1.3) with some

$$m > \frac{9}{8}, \quad (1.10)$$

then problem (1.1) possesses a global weak solution, which extends the results of Tao and Winkler ([15]), who showed the global existence of solutions in the cases $S(x, n, c) := C_S$, m satisfies (1.4) with $m > \frac{8}{7}$.

2 Preliminaries and main results

Due to the hypothesis (1.4), the problem (1.1) has no classical solutions in general, and thus we consider its weak solutions in the following sense.

Definition 2.1. (weak solutions) Let $T \in (0, \infty)$ and

$$H(s) = \int_0^s D(\sigma) d\sigma \quad \text{for } s \geq 0.$$

Suppose that (n_0, c_0, u_0) satisfies (2.7). Then a triple of functions (n, c, u) defined in $\Omega \times (0, T)$ is called a weak solution of model (1.1), if

$$\begin{cases} n \in L_{loc}^1(\bar{\Omega} \times [0, T)), \\ c \in L_{loc}^\infty(\bar{\Omega} \times [0, T)) \cap L_{loc}^1([0, T); W^{1,1}(\Omega)), \\ u \in L_{loc}^1([0, T); W^{1,1}(\Omega)), \end{cases} \quad (2.1)$$

where $n \geq 0$ and $c \geq 0$ in $\Omega \times (0, T)$ as well as $\nabla \cdot u = 0$ in the distributional sense in $\Omega \times (0, T)$, in addition,

$$H(n), \quad n|\nabla c| \quad \text{and} \quad n|u| \quad \text{belong to} \quad L_{loc}^1(\bar{\Omega} \times [0, T)), \quad (2.2)$$

and

$$\begin{aligned} - \int_0^T \int_{\Omega} n \varphi_t - \int_{\Omega} n_0 \varphi(\cdot, 0) &= \int_0^T \int_{\Omega} H(n) \Delta \varphi + \int_0^T \int_{\Omega} n (S(x, n, c) \cdot \nabla c) \cdot \nabla \varphi \\ &\quad + \int_0^T \int_{\Omega} n u \cdot \nabla \varphi \end{aligned} \quad (2.3)$$

for any $\varphi \in C_0^\infty(\bar{\Omega} \times [0, T))$ satisfying $\frac{\partial \varphi}{\partial \nu} = 0$ on $\partial\Omega \times (0, T)$ as well as

$$- \int_0^T \int_{\Omega} c \varphi_t - \int_{\Omega} c_0 \varphi(\cdot, 0) = - \int_0^T \int_{\Omega} \nabla c \cdot \nabla \varphi - \int_0^T \int_{\Omega} n c \cdot \varphi + \int_0^T \int_{\Omega} c u \cdot \nabla \varphi \quad (2.4)$$

for any $\varphi \in C_0^\infty(\bar{\Omega} \times [0, T))$ and

$$- \int_0^T \int_{\Omega} u \varphi_t - \int_{\Omega} u_0 \varphi(\cdot, 0) = - \int_0^T \int_{\Omega} \nabla u \cdot \nabla \varphi - \int_0^T \int_{\Omega} n \nabla \phi \cdot \varphi \quad (2.5)$$

for any $\varphi \in C_0^\infty(\bar{\Omega} \times [0, T); \mathbb{R}^3)$ fulfilling $\nabla \varphi \equiv 0$ in $\Omega \times (0, T)$. If (n, c, u) is a weak solution of (1.1) in $\Omega \times (0, T)$ for any $T \in (0, \infty)$, then we call (n, c, u) a global weak solution.

In this paper, we assume that

$$\phi \in W^{1,\infty}(\Omega). \quad (2.6)$$

Moreover, let the initial data (n_0, c_0, u_0) fulfill

$$\begin{cases} n_0 \in C^\kappa(\bar{\Omega}) \quad \text{for certain } \kappa > 0 \quad \text{with } n_0 \geq 0 \quad \text{in } \Omega, \\ c_0 \in W^{1,\infty}(\Omega) \quad \text{with } c_0 \geq 0 \quad \text{in } \bar{\Omega}, \\ u_0 \in D(A_r^\gamma) \quad \text{for some } \gamma \in (\frac{3}{4}, 1) \quad \text{and any } r \in (1, \infty), \end{cases} \quad (2.7)$$

where A_r denotes the Stokes operator with domain $D(A_r) := W^{2,r}(\Omega) \cap W_0^{1,r}(\Omega) \cap L_\sigma^r(\Omega)$, and $L_\sigma^r(\Omega) := \{\varphi \in L^r(\Omega) | \nabla \cdot \varphi = 0\}$ for $r \in (1, \infty)$ ([12]).

Theorem 2.1. *Let (2.6) hold, and suppose that m and S satisfies (1.4) and (1.2)–(1.3), respectively. Suppose that the assumptions (2.7) hold. If*

$$m > \frac{9}{8}, \quad (2.8)$$

then there exists at least one global weak solution (in the sense of Definition 2.1 above) of problem (1.1). Moreover, for any fixed $T > 0$, there exists a positive constant $C := C(T)$ such that

$$\|u(\cdot, t)\|_{L^\infty(\Omega)} + \|c(\cdot, t)\|_{W^{1,\infty}(\Omega)} + \|n(\cdot, t)\|_{L^\infty(\Omega)} \leq C \quad \text{for all } t \in (0, T). \quad (2.9)$$

Remark 2.1. (i) If $S(x, n, c) := C_S$, Theorem 2.1 extends the results of Theorem 1.1 of Tao and Winkler [15], who proved the possibility of **global existence**, in the case that $m > \frac{8}{7}$.

(ii) In view of Theorem 2.1, if the flow of fluid is ignored or the fluid is stationary in (1.1), $S(x, n, c) := C_S$, and $N = 3$, Theorem 2.1 is consistent with the result of Theorem 2.1 of Zheng and Wang([28]), who proved the possibility of **global existence**, in the case that $m > \frac{9}{8}$.

(iii) From Theorem 2.1, if the flow of fluid is ignored or the fluid is stationary in (1.1) and $S(x, n, c) := C_S$, and $N = 3$, our results improve of Wang et al. [17], who proved the possibility of **global existence**, in the case that $m > \frac{4}{3}$.

(iv) By Theorem 2.1, we also derive that the large diffusion exponent $m (> \frac{10}{9})$ yields the existence of solutions to (1.1). Moreover, no smallness condition on either ϕ or on the initial data needs to be fulfilled here, which is different from [3].

Lemma 2.1. ([22]) *Let $l \in [1, +\infty)$ and $r \in [1, +\infty]$ be such that*

$$\begin{cases} l < \frac{3r}{3-r} & \text{if } r \leq 3, \\ l \leq \infty & \text{if } r > 3. \end{cases} \quad (2.10)$$

Then for all $K > 0$ there exists $C := C(l, r, K)$ such that if

$$\|n(\cdot, t)\|_{L^r(\Omega)} \leq K \quad \text{for all } t \in (0, T_{\max}), \quad (2.11)$$

then

$$\|Du(\cdot, t)\|_{L^1(\Omega)} \leq C \text{ for all } t \in (0, T_{max}). \quad (2.12)$$

In general, the degenerate diffusion case of (1.1) might not have classical solutions, thus in order to justify all the formal arguments, we need to introduce the following approximating system of (1.1):

$$\left\{ \begin{array}{l} n_{\varepsilon t} + u_{\varepsilon} \cdot \nabla n_{\varepsilon} = \nabla \cdot (D_{\varepsilon}(n_{\varepsilon}) \nabla n_{\varepsilon}) - \nabla \cdot (n_{\varepsilon} S_{\varepsilon}(x, n_{\varepsilon}, c_{\varepsilon}) \nabla c_{\varepsilon}), \quad x \in \Omega, t > 0, \\ c_{\varepsilon t} + u_{\varepsilon} \cdot \nabla c_{\varepsilon} = \Delta c_{\varepsilon} - n_{\varepsilon} c_{\varepsilon}, \quad x \in \Omega, t > 0, \\ u_{\varepsilon t} + \nabla P_{\varepsilon} = \Delta u_{\varepsilon} + n_{\varepsilon} \nabla \phi, \quad x \in \Omega, t > 0, \\ \nabla \cdot u_{\varepsilon} = 0, \quad x \in \Omega, t > 0, \\ \nabla n_{\varepsilon} \cdot \nu = \nabla c_{\varepsilon} \cdot \nu = 0, u_{\varepsilon} = 0, \quad x \in \partial\Omega, t > 0, \\ n_{\varepsilon}(x, 0) = n_0(x), c_{\varepsilon}(x, 0) = c_0(x), u_{\varepsilon}(x, 0) = u_0(x), \quad x \in \Omega, \end{array} \right. \quad (2.13)$$

where a family $(D_{\varepsilon})_{\varepsilon \in (0,1)}$ of functions

$$D_{\varepsilon} \in C^2((0, \infty)) \text{ such that } D_{\varepsilon}(n) \geq \varepsilon \text{ for all } n > 0$$

$$\text{and } D(n) \leq D_{\varepsilon}(n) \leq D(n) + 2\varepsilon \text{ for all } n > 0 \text{ and } \varepsilon \in (0, 1),$$

$$S_{\varepsilon}(x, n, c) := \rho_{\varepsilon}(x) S(x, n, c), \quad x \in \bar{\Omega}, \quad n \geq 0, \quad c \geq 0 \quad \text{and} \quad \varepsilon \in (0, 1). \quad (2.14)$$

Here $(\rho_{\varepsilon})_{\varepsilon \in (0,1)} \in C_0^{\infty}(\Omega)$ be a family of standard cut-off functions satisfying $0 \leq \rho_{\varepsilon} \leq 1$ in Ω and $\rho_{\varepsilon} \rightarrow 1$ in Ω as $\varepsilon \rightarrow 0$.

Let us begin with the following statement on local well-posedness of (2.13), along with a convenient extensibility criterion. For a proof we refer to (see [22], Lemma 2.1 of [21]):

Lemma 2.2. *Let $\Omega \subseteq \mathbb{R}^3$ be a bounded convex domain with smooth boundary. Assume that the initial data (n_0, c_0, u_0) fulfills (2.7). Then there exist $T_{max} \in (0, \infty]$ and a classical solution $(n_{\varepsilon}, c_{\varepsilon}, u_{\varepsilon}, P_{\varepsilon})$ of (2.13) in $\Omega \times (0, T_{max})$ such that*

$$\left\{ \begin{array}{l} n_{\varepsilon} \in C^0(\bar{\Omega} \times [0, T_{max})) \cap C^{2,1}(\bar{\Omega} \times (0, T_{max})), \\ c_{\varepsilon} \in C^0(\bar{\Omega} \times [0, T_{max})) \cap C^{2,1}(\bar{\Omega} \times (0, T_{max})), \\ u_{\varepsilon} \in C^0(\bar{\Omega} \times [0, T_{max})) \cap C^{2,1}(\bar{\Omega} \times (0, T_{max})), \\ P_{\varepsilon} \in C^{1,0}(\bar{\Omega} \times (0, T_{max})), \end{array} \right. \quad (2.15)$$

classically solving (2.13) in $\Omega \times [0, T_{max})$. Moreover, n_ε and c_ε are nonnegative in $\Omega \times (0, T_{max})$, and

$$\limsup_{t \nearrow T_{max}} (\|n_\varepsilon(\cdot, t)\|_{L^\infty(\Omega)} + \|c_\varepsilon(\cdot, t)\|_{W^{1,\infty}(\Omega)} + \|A^\gamma u_\varepsilon(\cdot, t)\|_{L^2(\Omega)}) = \infty, \quad (2.16)$$

where γ is given by (2.7).

Lemma 2.3. ([9]) Let $w \in C^2(\bar{\Omega})$ satisfy $\nabla w \cdot \nu = 0$ on $\partial\Omega$.

(i) Then

$$\frac{\partial |\nabla w|^2}{\partial \nu} \leq C_{\partial\Omega} |\nabla w|^2,$$

where $C_{\partial\Omega}$ is an upper bound on the curvature of $\partial\Omega$.

(ii) Furthermore, for any $\delta > 0$ there is $C(\delta) > 0$ such that every $w \in C^2(\bar{\Omega})$ with $\nabla w \cdot \nu = 0$ on $\partial\Omega$ fulfils

$$\|w\|_{L^2(\partial\Omega)} \leq \delta \|\Delta w\|_{L^2(\Omega)} + C(\delta) \|w\|_{L^2(\Omega)}.$$

(iii) For any positive $w \in C^2(\bar{\Omega})$

$$\|\Delta w^{\frac{1}{2}}\|_{L^2(\Omega)} \leq \frac{1}{2} \|w^{\frac{1}{2}} \Delta \ln w\|_{L^2(\Omega)} + \frac{1}{4} \|w^{-\frac{3}{2}} |\nabla w|^2\|_{L^2(\Omega)}. \quad (2.17)$$

(iv) There are $C > 0$ and $\delta > 0$ such that every positive $w \in C^2(\bar{\Omega})$ fulfilling $\nabla w \cdot \nu = 0$ on $\partial\Omega$ satisfies

$$-2 \int_{\Omega} \frac{|\Delta w|^2}{w} + \int_{\Omega} \frac{|\nabla w|^2 \Delta w}{w^2} \leq -\delta \int_{\Omega} w |D^2 \ln w|^2 - \delta \int_{\Omega} \frac{|\nabla w|^4}{w^3} + C \int_{\Omega} w. \quad (2.18)$$

Lemma 2.4. (Lemma 3.8 of [22]) Let $q \geq 1$,

$$\lambda \in [2q + 2, 4q + 1] \quad (2.19)$$

and $\Omega \subset \mathbb{R}^3$ be a bounded convex domain with smooth boundary. Then there exists $C > 0$ such that for all $\varphi \in C^2(\bar{\Omega})$ fulfilling $\varphi \cdot \frac{\partial \varphi}{\partial \nu} = 0$ on $\partial\Omega$ we have

$$\|\nabla \varphi\|_{L^\lambda(\Omega)} \leq C \| |\nabla \varphi|^{q-1} D^2 \varphi \|_{L^2(\Omega)}^{\frac{2(\lambda-3)}{(2q-1)\lambda}} \|\varphi\|_{L^\infty(\Omega)}^{\frac{6q-\lambda}{(2q-1)\lambda}} + C \|\varphi\|_{L^\infty(\Omega)}. \quad (2.20)$$

Let us state two well-known results of solution of (2.13).

Lemma 2.5. *The solution of (2.13) satisfies*

$$\|n_\varepsilon(\cdot, t)\|_{L^1(\Omega)} = \|n_0\|_{L^1(\Omega)} \quad \text{for all } t \in (0, T_{\max}) \quad (2.21)$$

and

$$\|c_\varepsilon(\cdot, t)\|_{L^\infty(\Omega)} \leq \|c_0\|_{L^\infty(\Omega)} \quad \text{for all } t \in (0, T_{\max}). \quad (2.22)$$

Lemma 2.6. *For any $l < \frac{3}{2}$, there exists $C := C(l, \|n_0\|_{L^1(\Omega)})$ such that*

$$\|Du(\cdot, t)\|_{L^l(\Omega)} \leq C \quad \text{for all } t \in (0, T_{\max}). \quad (2.23)$$

Proof. Choosing $r = 1$ in Lemma 2.7 and using (2.21), we can get the results. \square

Lemma 2.7. *Let $m > \frac{10}{9}$. There exists $C > 0$ independent of ε such that for every $\delta_1 > 0$, the solution of (2.13) satisfies*

$$\int_{\Omega} |u_\varepsilon|^2 + \int_{\Omega} |\nabla u_\varepsilon|^2 \leq \delta_1 \int_{\Omega} \frac{D_\varepsilon(n_\varepsilon) |\nabla n_\varepsilon|^2}{n_\varepsilon} + C \quad \text{for all } t \in (0, T_{\max}), \quad (2.24)$$

Proof. Testing the third equation of (2.13) with u_ε , integrating by parts and using $\nabla \cdot u_\varepsilon = 0$

$$\frac{1}{2} \int_{\Omega} |u_\varepsilon|^2 + \int_{\Omega} |\nabla u_\varepsilon|^2 = \int_{\Omega} n_\varepsilon u_\varepsilon \cdot \nabla \phi \quad \text{for all } t \in (0, T_{\max}), \quad (2.25)$$

which together with the Hölder inequality, (2.6), the continuity of the embedding $W^{1,2}(\Omega) \hookrightarrow L^6(\Omega)$, the Gagliardo–Nirenberg inequality and (2.21) implies that there exists a positive constants C_1, C_2 and C_3 independent of ε such that

$$\begin{aligned} \int_{\Omega} n_\varepsilon u_\varepsilon \cdot \nabla \phi &\leq \|\nabla \phi\|_{L^\infty(\Omega)} \|n_\varepsilon\|_{L^{\frac{6}{5}}(\Omega)} \|\nabla u_\varepsilon\|_{L^2(\Omega)} \\ &\leq C_1 \|n_\varepsilon\|_{L^{\frac{6}{5}}(\Omega)} \|\nabla u_\varepsilon\|_{L^2(\Omega)} \\ &\leq C_2 \|\nabla n_\varepsilon^{\frac{m}{2}}\|_{L^2(\Omega)}^{\frac{1}{3m-1}} \|n_\varepsilon^{\frac{m}{2}}\|_{L^{\frac{2}{m}}(\Omega)}^{\frac{2}{m} - \frac{1}{3m-1}} \|\nabla u_\varepsilon\|_{L^2(\Omega)} \\ &\leq C_3 (\|\nabla n_\varepsilon^{\frac{m}{2}}\|_{L^2(\Omega)}^{\frac{1}{3m-1}} + 1) \|\nabla u_\varepsilon\|_{L^2(\Omega)} \quad \text{for all } t \in (0, T_{\max}). \end{aligned} \quad (2.26)$$

Next, with the help of the Young inequality and $m > \frac{10}{9}$, inserting (2.26) into (2.25) and (1.4), we derive that

$$\begin{aligned} \frac{1}{2} \int_{\Omega} |u_\varepsilon|^2 + \frac{1}{2} \int_{\Omega} |\nabla u_\varepsilon|^2 &\leq C_4 (\|\nabla n_\varepsilon^{\frac{m}{2}}\|_{L^2(\Omega)}^{\frac{2}{3m-1}} + 1) \\ &\leq \frac{\delta_1}{2C_D} \|\nabla n_\varepsilon^{\frac{m}{2}}\|_{L^2(\Omega)}^{\frac{2}{3m-1}} + C_5 \\ &\leq \frac{\delta_1}{2} \int_{\Omega} \frac{D_\varepsilon(n_\varepsilon) |\nabla n_\varepsilon|^2}{n_\varepsilon} + C_5 \quad \text{for all } t \in (0, T_{\max}), \end{aligned} \quad (2.27)$$

and some positive constants C_4 and C_5 . \square

Lemma 2.8. *Let $\frac{10}{9} < m \leq 2$. There exist μ_0 and $C > 0$ independent of ε such that for every $\delta_i (i = 2, 3, 4, 5) > 0$*

$$\begin{aligned} & \frac{d}{dt} \int_{\Omega} \frac{|\nabla c_{\varepsilon}|^2}{c_{\varepsilon}} + \mu_0 \int_{\Omega} c_{\varepsilon} |D^2 \ln c_{\varepsilon}|^2 + \left(\mu_0 - \frac{\delta_2}{4} - \frac{\delta_3}{4}\right) \int_{\Omega} \frac{|\nabla c_{\varepsilon}|^4}{c_{\varepsilon}^3} \\ & \leq \left(\frac{\delta_4}{4} + \frac{\delta_5}{4}\right) \int_{\Omega} \frac{D_{\varepsilon}(n_{\varepsilon}) |\nabla n_{\varepsilon}|^2}{n_{\varepsilon}} + \frac{4}{\delta_2} \|c_0\|_{L^{\infty}(\Omega)} \int_{\Omega} |\nabla u_{\varepsilon}|^2 + C \quad \text{for all } t \in (0, T_{\max}) \end{aligned} \quad (2.28)$$

Proof. Firstly, by calculation, we derive that

$$\begin{aligned} \frac{d}{dt} \int_{\Omega} \frac{|\nabla c_{\varepsilon}|^2}{c_{\varepsilon}} &= 2 \int_{\Omega} \frac{\nabla c_{\varepsilon} \cdot \nabla c_{\varepsilon t}}{c_{\varepsilon}} - \int_{\Omega} \frac{|\nabla c_{\varepsilon}|^2 c_{\varepsilon t}}{c_{\varepsilon}^2} \\ &= -2 \int_{\Omega} \frac{\Delta c_{\varepsilon} c_{\varepsilon t}}{c_{\varepsilon}} + \int_{\Omega} \frac{|\nabla c_{\varepsilon}|^2 c_{\varepsilon t}}{c_{\varepsilon}^2} \\ &= -2 \int_{\Omega} \frac{|\Delta c_{\varepsilon}|^2}{c_{\varepsilon}} + 2 \int_{\Omega} \frac{\Delta c_{\varepsilon} n_{\varepsilon} c_{\varepsilon}}{c_{\varepsilon}} + 2 \int_{\Omega} \frac{\Delta c_{\varepsilon}}{c_{\varepsilon}} u_{\varepsilon} \cdot \nabla c_{\varepsilon} \\ &\quad + \int_{\Omega} \frac{|\nabla c_{\varepsilon}|^2 \Delta c_{\varepsilon}}{c_{\varepsilon}^2} - \int_{\Omega} \frac{|\nabla c_{\varepsilon}|^2 n_{\varepsilon} c_{\varepsilon}}{c_{\varepsilon}^2} - \int_{\Omega} \frac{|\nabla c_{\varepsilon}|^2 u_{\varepsilon} \cdot \nabla c_{\varepsilon}}{c_{\varepsilon}^2} \quad \text{for all } t \in (0, T_{\max}). \end{aligned} \quad (2.29)$$

Due to (vi) of Lemma 2.3 and the Young inequality, there exist $\mu_0 > 0$ and $C(\mu_0) > 0$ such that

$$-2 \int_{\Omega} \frac{|\Delta c_{\varepsilon}|^2}{c_{\varepsilon}} + \int_{\Omega} \frac{|\nabla c_{\varepsilon}|^2 \Delta c_{\varepsilon}}{c_{\varepsilon}^2} \leq -\mu_0 \int_{\Omega} c_{\varepsilon} |D^2 \ln c_{\varepsilon}|^2 - \mu_0 \int_{\Omega} \frac{|\nabla c_{\varepsilon}|^4}{c_{\varepsilon}^3} + C(\mu_0) \int_{\Omega} c_{\varepsilon} \quad (2.30)$$

for all $t \in (0, T_{\max})$. On the other hand, for all $t \in (0, T_{\max})$, with the help of the computing, the Young inequality and Lemma 2.5 implies that for any $\delta_2 > 0$

$$\begin{aligned} & 2 \int_{\Omega} \frac{\Delta c_{\varepsilon}}{c_{\varepsilon}} (u_{\varepsilon} \cdot \nabla c_{\varepsilon}) - \int_{\Omega} \frac{|\nabla c_{\varepsilon}|^2}{c_{\varepsilon}^2} u_{\varepsilon} \cdot \nabla c_{\varepsilon} \\ &= 2 \int_{\Omega} \frac{|\nabla c_{\varepsilon}|^2}{c_{\varepsilon}^2} u_{\varepsilon} \cdot \nabla c_{\varepsilon} - 2 \int_{\Omega} \frac{1}{c_{\varepsilon}} \nabla c_{\varepsilon} \cdot (\nabla u_{\varepsilon} \nabla c_{\varepsilon}) - 2 \int_{\Omega} \frac{1}{c_{\varepsilon}} u_{\varepsilon} \cdot D^2 c_{\varepsilon} \nabla c_{\varepsilon} + 2 \int_{\Omega} \frac{1}{c_{\varepsilon}} u_{\varepsilon} \cdot D^2 c_{\varepsilon} \nabla c_{\varepsilon} \\ &= -2 \int_{\Omega} \frac{1}{c_{\varepsilon}} \nabla c_{\varepsilon} \cdot (\nabla u_{\varepsilon} \nabla c_{\varepsilon}) \\ &\leq \frac{\delta_2}{4} \int_{\Omega} \frac{|\nabla c_{\varepsilon}|^4}{c_{\varepsilon}^3} + \frac{4}{\delta_2} \int_{\Omega} c_{\varepsilon} |\nabla u_{\varepsilon}|^2 \\ &\leq \frac{\delta_2}{4} \int_{\Omega} \frac{|\nabla c_{\varepsilon}|^4}{c_{\varepsilon}^3} + C_1 \int_{\Omega} |\nabla u_{\varepsilon}|^2 \quad \text{for all } t \in (0, T_{\max}), \end{aligned} \quad (2.31)$$

where $C_1 := \frac{4}{\delta_2} \|c_0\|_{L^{\infty}(\Omega)}$. In view of integration by parts, the Young inequality, (1.4) and

(2.22), we also derive that

$$\begin{aligned}
2 \int_{\Omega} \frac{\Delta c_{\varepsilon} n_{\varepsilon} c_{\varepsilon}}{c_{\varepsilon}} &= -2 \int_{\Omega} \nabla n_{\varepsilon} \cdot \nabla c_{\varepsilon} \\
&\leq \frac{\delta_3}{4} \int_{\Omega} \frac{|\nabla c_{\varepsilon}|^4}{c_{\varepsilon}^3} + 2^{\frac{4}{3}} \delta_3^{-\frac{1}{3}} \int_{\Omega} c_{\varepsilon} |\nabla n_{\varepsilon}|^{\frac{4}{3}} \\
&\leq \frac{\delta_3}{4} \int_{\Omega} \frac{|\nabla c_{\varepsilon}|^4}{c_{\varepsilon}^3} + \frac{\delta_4}{4C_D} \int_{\Omega} n_{\varepsilon}^{m-2} |\nabla n_{\varepsilon}|^2 + C_2 \int_{\Omega} n_{\varepsilon}^{4-2m} c_{\varepsilon}^3 \\
&\leq \frac{\delta_3}{4} \int_{\Omega} \frac{|\nabla c_{\varepsilon}|^4}{c_{\varepsilon}^3} + \frac{\delta_4}{4} \int_{\Omega} \frac{D_{\varepsilon}(n_{\varepsilon}) |\nabla n_{\varepsilon}|^2}{n_{\varepsilon}} + C_3 \int_{\Omega} n_{\varepsilon}^{4-2m} \quad \text{for all } t \in (0, T_{max}),
\end{aligned} \tag{2.32}$$

where $\delta_3, \delta_4, C_2 := C_2(\delta_3, \delta_4), C_3 := C_3(\delta_3, \delta_4, \|c_0\|_{L^\infty(\Omega)})$ are positive constants.

Case $\frac{10}{9} < m < \frac{3}{2}$: Due to the Gagliardo–Nirenberg inequality and (2.21), we conclude that

$$\begin{aligned}
C_3 \int_{\Omega} n_{\varepsilon}^{4-2m} &= C_3 \|n_{\varepsilon}^{\frac{m}{2}}\|_{L^{\frac{2(4-2m)}{m}}(\Omega)}^{\frac{2(4-2m)}{m}} \\
&\leq C_4 \|\nabla n_{\varepsilon}^{\frac{m}{2}}\|_{L^2(\Omega)}^{\frac{2(4-2m)\mu_1}{m}} \|n_{\varepsilon}^{\frac{m}{2}}\|_{L^{\frac{2}{m}}(\Omega)}^{\frac{2(4-2m)(1-\mu_1)}{m}} + \|n_{\varepsilon}^{\frac{m}{2}}\|_{L^{\frac{2}{m}}(\Omega)}^{\frac{2(4-2m)}{m}} \\
&\leq C_5 (\|\nabla n_{\varepsilon}^{\frac{m}{2}}\|_{L^2(\Omega)}^{\frac{2(4-2m)\mu_1}{m}} + 1) \\
&= C_5 (\|\nabla n_{\varepsilon}^{\frac{m}{2}}\|_{L^2(\Omega)}^{\frac{6(3-2m)}{3m-1}} + 1) \quad \text{for all } t \in (0, T_{max}),
\end{aligned} \tag{2.33}$$

where C_4 and C_5 are positive constants,

$$\mu_1 = \frac{\frac{3m}{2} - \frac{3m}{2(4-2m)}}{\frac{3m-1}{2}} \in (0, 1).$$

Now, in view of $m > \frac{10}{9}$, with the help of the Young inequality and (2.33), for any $\delta_5 > 0$, we have

$$C_3 \int_{\Omega} n_{\varepsilon}^{4-2m} \leq \frac{\delta_5}{4C_D} \|\nabla n_{\varepsilon}^{\frac{m}{2}}\|_{L^2(\Omega)}^2 + C_6 \quad \text{for all } t \in (0, T_{max}) \tag{2.34}$$

with some $C_6 > 0$.

Case $\frac{3}{2} \leq m \leq 2$: With the help of the Young inequality and (2.21), we derive that

$$\begin{aligned}
C_3 \int_{\Omega} n_{\varepsilon}^{4-2m} &\leq \int_{\Omega} n_{\varepsilon} + C_7 \\
&\leq C_8 \quad \text{for all } t \in (0, T_{max})
\end{aligned} \tag{2.35}$$

where C_7 and C_8 are positive constants independent of ε . Finally, collecting (2.30)–(2.35) and (2.29), we can get the results. \square

Lemma 2.9. *Let $\frac{10}{9} < m \leq 2$ and $\delta > 0$. There is $C > 0$ such that for any δ_6 and δ_7*

$$\frac{d}{dt} \int_{\Omega} n_{\varepsilon} \ln n_{\varepsilon} + (1 - \frac{\delta_7}{4}) \int_{\Omega} \frac{D_{\varepsilon}(n_{\varepsilon}) |\nabla n_{\varepsilon}|^2}{n_{\varepsilon}} \leq \frac{\delta_6}{4} \int_{\Omega} \frac{|\nabla c_{\varepsilon}|^4}{c_{\varepsilon}^3} + C \quad \text{for all } t \in (0, T_{max}). \tag{2.36}$$

Proof. Using these estimates and the first equation of (2.13), from integration by parts we obtain

$$\begin{aligned}
\frac{d}{dt} \int_{\Omega} n_{\varepsilon} \ln n_{\varepsilon} &= \int_{\Omega} n_{\varepsilon t} \ln n_{\varepsilon} + \int_{\Omega} n_{\varepsilon t} \\
&= \int_{\Omega} \nabla \cdot (D_{\varepsilon}(n_{\varepsilon}) \nabla n_{\varepsilon}) \ln n_{\varepsilon} - \int_{\Omega} \ln n_{\varepsilon} \nabla \cdot (n_{\varepsilon} S_{\varepsilon}(x, n_{\varepsilon}, c_{\varepsilon}) \nabla c_{\varepsilon}) - \int_{\Omega} \ln n_{\varepsilon} u_{\varepsilon} \cdot \nabla n_{\varepsilon} \\
&\leq - \int_{\Omega} \frac{D_{\varepsilon}(n_{\varepsilon}) |\nabla n_{\varepsilon}|^2}{n_{\varepsilon}} + \int_{\Omega} S_0(c_{\varepsilon}) |\nabla n_{\varepsilon}| |\nabla c_{\varepsilon}|
\end{aligned} \tag{2.37}$$

for all $t \in (0, T_{max})$. Now, in view of (2.22), employing the same argument of (2.32)–(2.35), for any $\delta_6 > 0$ and $\delta_7 > 0$, we conclude that

$$\begin{aligned}
&\int_{\Omega} S_0(c_{\varepsilon}) |\nabla n_{\varepsilon}| |\nabla c_{\varepsilon}| \\
&\leq S_0(\|c_0\|_{L^{\infty}(\Omega)}) \int_{\Omega} |\nabla n_{\varepsilon}| |\nabla c_{\varepsilon}| \\
&\leq \frac{\delta_6}{4} \int_{\Omega} \frac{|\nabla c_{\varepsilon}|^4}{c_{\varepsilon}^3} + \frac{\delta_7}{4} \int_{\Omega} \frac{D_{\varepsilon}(n_{\varepsilon}) |\nabla n_{\varepsilon}|^2}{n_{\varepsilon}} + C_9 \quad \text{for all } t \in (0, T_{max}).
\end{aligned} \tag{2.38}$$

with $C_9 > 0$ independent of ε . Now, in conjunction with (2.37) and (2.38), we get the results.

This completes the proof of Lemma 2.9. \square

Properly combining Lemmata 2.7–2.9, we arrive at the following Lemma, which plays a key rule in obtaining the existence of solutions to (2.13).

Lemma 2.10. *Let $\frac{10}{9} < m \leq 2$ and S satisfy (1.2)–(1.3). Suppose that (1.4) and (2.6)–(2.7) holds. Then there exists $C > 0$ such that the solution of (2.13) satisfies*

$$\int_{\Omega} n_{\varepsilon}(\cdot, t) \ln n_{\varepsilon}(\cdot, t) + \int_{\Omega} |\nabla \sqrt{c_{\varepsilon}}(\cdot, t)|^2 + \int_{\Omega} |u_{\varepsilon}(\cdot, t)|^2 \leq C \tag{2.39}$$

for all $t \in (0, T_{max})$. Moreover, for each $T \in (0, T_{max})$, one can find a constant $C > 0$ such that

$$\int_0^T \int_{\Omega} n_{\varepsilon}^{m-2} |\nabla n_{\varepsilon}|^2 \leq C \tag{2.40}$$

and

$$\int_0^T \int_{\Omega} |\nabla c_{\varepsilon}|^4 \leq C \tag{2.41}$$

as well as

$$\int_0^T \int_{\Omega} c_{\varepsilon} |D^2 \ln c_{\varepsilon}|^2 \leq C. \tag{2.42}$$

Proof. Take an evident linear combination of the inequalities provided by Lemmata 2.7–2.9, we conclude that there exists a positive constant $C_1 > 0$ such that

$$\begin{aligned} & \frac{d}{dt} \left(\int_{\Omega} \frac{|\nabla c_{\varepsilon}|^2}{c_{\varepsilon}} + L \int_{\Omega} n_{\varepsilon} \ln n_{\varepsilon} + K |u_{\varepsilon}|^2 \right) + \left(K - \frac{4}{\delta_2} \|c_0\|_{L^{\infty}(\Omega)} \right) \int_{\Omega} |\nabla u_{\varepsilon}|^2 + \mu_0 \int_{\Omega} c_{\varepsilon} |D^2 \ln c_{\varepsilon}|^2 \\ & + \left[\left(\mu_0 - \frac{\delta_2}{4} - \frac{\delta_3}{4} \right) - L \frac{\delta_6}{4} \right] \int_{\Omega} \frac{|\nabla c_{\varepsilon}|^4}{c_{\varepsilon}^3} + \left[L \left(1 - \frac{\delta_7}{4} \right) - \frac{\delta_4}{4} - \frac{\delta_5}{4} - K \delta_1 \right] \int_{\Omega} \frac{D_{\varepsilon}(n_{\varepsilon}) |\nabla n_{\varepsilon}|^2}{n_{\varepsilon}} \\ & \leq C_1 \quad \text{for all } t \in (0, T_{\max, \varepsilon}), \end{aligned} \tag{2.43}$$

where K, L are positive constants. Now, choosing $\delta_7 = 1$, $\delta_6 = \frac{\mu_0}{L}$, $\delta_3 = \mu_0$, $\delta_4 = \delta_5 = L$, $\delta_1 = \frac{L}{8K}$ and $\delta_2 = \frac{8}{K} \|c_0\|_{L^{\infty}(\Omega)}$ and K large enough such that $\frac{8}{K} \|c_0\|_{L^{\infty}(\Omega)} < \mu_0$ in (2.43), we derive that

$$\begin{aligned} & \frac{d}{dt} \left(\int_{\Omega} \frac{|\nabla c_{\varepsilon}|^2}{c_{\varepsilon}} + L \int_{\Omega} n_{\varepsilon} \ln n_{\varepsilon} + K |u_{\varepsilon}|^2 \right) + \frac{K}{2} \int_{\Omega} |\nabla u_{\varepsilon}|^2 + \mu_0 \int_{\Omega} c_{\varepsilon} |D^2 \ln c_{\varepsilon}|^2 \\ & + \frac{\mu_0}{4} \int_{\Omega} \frac{|\nabla c_{\varepsilon}|^4}{c_{\varepsilon}^3} + \frac{L}{8} \int_{\Omega} \frac{D_{\varepsilon}(n_{\varepsilon}) |\nabla n_{\varepsilon}|^2}{n_{\varepsilon}} \\ & \leq C_2 \quad \text{for all } t \in (0, T_{\max, \varepsilon}). \end{aligned} \tag{2.44}$$

and some positive constant C_2 . Hence, by some basic calculation, we can conclude (2.39)–(2.42). \square

Remark 2.2. From Lemma 2.10, $m > \frac{10}{9}$ yields the (2.39), which is the natural energy functional of 3D chemotaxis-stokes system with nonlinear diffusion and rotation. Furthermore, Lemma 2.10 expands the contents of [22] (see also the introduction of [2], Wang et. al [18, 19]). Moreover, rely on Lemma 2.10, we also conclude that the large diffusion exponent $m (> \frac{10}{9})$ benefits the existence of solutions to (2.13).

Employing almost exactly the same arguments as in the proof of Lemma 3.5 and Lemma 3.6 in [15] (the minor necessary changes are left as an easy exercise to the reader), we conclude the following Lemmata:

Lemma 2.11. *Let $p > 1$. Then the solution of (2.13) from Lemma 2.2 satisfies*

$$\frac{1}{p} \frac{d}{dt} \|n_{\varepsilon}\|_{L^p(\Omega)}^p + \frac{C_D(p-1)}{2} \int_{\Omega} n_{\varepsilon}^{m+p-3} |\nabla n_{\varepsilon}|^2 \leq C \int_{\Omega} n_{\varepsilon}^{p+1-m} |\nabla c_{\varepsilon}|^2, \tag{2.45}$$

where $C > 0$ is a positive constant depends on p and C_D .

Lemma 2.12. *Let $q > 1$. Then the solution of (2.13) from Lemma 2.2 satisfies*

$$\begin{aligned} & \frac{1}{2q} \frac{d}{dt} \|\nabla c_\varepsilon\|_{L^{2q}(\Omega)}^{2q} + \frac{(q-1)}{q^2} \int_{\Omega} |\nabla |\nabla c_\varepsilon|^q|^2 + \frac{1}{2} \int_{\Omega} |\nabla c_\varepsilon|^{2q-2} |D^2 c_\varepsilon|^2 \\ & \leq C \left(\int_{\Omega} n_\varepsilon^2 |\nabla c_\varepsilon|^{2q-2} + \int_{\Omega} |Du_\varepsilon| |\nabla c_\varepsilon|^{2q} \right), \end{aligned} \quad (2.46)$$

where $C > 0$ is a positive constant depends on q and $\|c_0\|_{L^\infty(\Omega)}$.

In the following, we will estimate the integrals on the right-hand sides of (2.45) and (2.46). Indeed, we first give the following Lemma which plays an important rule in estimating the boundedness of $|Du_\varepsilon| |\nabla c_\varepsilon|^{2q}$.

Lemma 2.13. *(Lemma 3.7 of [22]) For any $\varepsilon \in (0, 1)$, we have*

$$\frac{d}{dt} \int_{\Omega} |A^{\frac{1}{2}} u_\varepsilon|^2 + \int_{\Omega} |Au_\varepsilon|^2 \leq \|\nabla \phi\|_{L^\infty(\Omega)}^2 \int_{\Omega} n_\varepsilon^2 \quad \text{for all } t \in (0, T_{max}). \quad (2.47)$$

Now, in order to estimate $\int_{\Omega} n_\varepsilon^2 |\nabla c_\varepsilon|^{2q-2}$, we recall the following Lemma which comes from Lemma 2.7 of [15] (see also [28])

Lemma 2.14. *(Lemma 2.7 of [15]) Assume that the initial data (n_0, c_0, u_0) fulfills (2.7).*

Let $p_0 \in (0, 9(m-1))$, $m > 1$ and $T_0 > 0$, then one can find a constant $C(p_0, T) > 0$ such that

$$\|n_\varepsilon(\cdot, t)\|_{L^{p_0}(\Omega)} \leq C(p_0, T) \quad \text{for all } t \in (0, T) \quad (2.48)$$

holds with

$$T := \min\{T_0, T_{max}\}. \quad (2.49)$$

Lemma 2.15. *Assuming that $p_0 \in [1, 9(m-1))$, $m > 1$ and $q > 1$. If*

$$\max\left\{1, \frac{p_0}{3} + 1 - m, m - 1 + p_0 \frac{q}{q+1}\right\} < p < [2(m-1) + \frac{2p_0}{3}]q + m - 1, \quad (2.50)$$

then for all small $\delta > 0$, we can find a constant $C := C(p, q, \delta) > 0$ such that

$$\int_{\Omega} n_\varepsilon^{p+1-m} |\nabla c_\varepsilon|^2 \leq \delta \int_{\Omega} |n_\varepsilon^{\frac{p+m-1}{2}}|^2 + \delta \|\nabla c_\varepsilon\|_{L^2(\Omega)}^{q-1} \|D^2 c_\varepsilon\|_{L^2(\Omega)}^2 + C \quad (2.51)$$

for all $t \in (0, T)$, where T is given by (2.49).

Proof. We apply the Hölder inequality with exponents $q + 1$ and $\frac{q+1}{q}$ to obtain

$$\begin{aligned}
J_1 &:= \int_{\Omega} n_{\varepsilon}^{p+1-m} |\nabla c_{\varepsilon}|^2 \\
&\leq \left(\int_{\Omega} n_{\varepsilon}^{\frac{q+1}{q}(p+1-m)} \right)^{\frac{1}{q+1}} \left(\int_{\Omega} |\nabla c_{\varepsilon}|^{2(q+1)} \right)^{\frac{1}{q}} \\
&= \|n_{\varepsilon}^{\frac{m+p-1}{2}}\|_{L^{\frac{2(q+1)}{q}(p+1-m)}(\Omega)}^{\frac{2(p+1-m)}{m+p-1}} \|\nabla c_{\varepsilon}\|_{L^{2(q+1)}(\Omega)}^2.
\end{aligned} \tag{2.52}$$

Due to $p > m - 1 + p_0 \frac{q}{q+1}$, $m > 1$ and $q > 1$, we have

$$\frac{p_0}{m+p-1} \leq \frac{\frac{q+1}{q}(p+1-m)}{m+p-1} \leq 3,$$

which together with Lemma 2.14 and the Gagliardo–Nirenberg inequality (see e.g. [26])

implies that there exist positive constants C_1 and C_2 such that

$$\begin{aligned}
&\|n_{\varepsilon}^{\frac{m+p-1}{2}}\|_{L^{\frac{2(q+1)}{q}(p+1-m)}(\Omega)}^{\frac{2(p+1-m)}{m+p-1}} \\
&\leq C_1 \left(\|n_{\varepsilon}^{\frac{p+m-1}{2}}\|_{L^2(\Omega)}^{\mu_1} \|n_{\varepsilon}^{\frac{m+p-1}{2}}\|_{L^{\frac{2p_0}{m+p-1}}(\Omega)}^{1-\mu_1} + \|n_{\varepsilon}^{\frac{m+p-1}{2}}\|_{L^{\frac{2p_0}{m+p-1}}(\Omega)}^{\frac{2(p+1-m)}{m+p-1}} \right) \\
&\leq C_2 \left(\|n_{\varepsilon}^{\frac{p+m-1}{2}}\|_{L^2(\Omega)}^{\frac{2(p+1-m)\mu_1}{m+p-1}} + 1 \right) \\
&= C_2 \left(\|n_{\varepsilon}^{\frac{p+m-1}{2}}\|_{L^2(\Omega)}^{\frac{3(\frac{p+1-m}{p_0} - \frac{q}{q+1})}{-\frac{1}{2} + \frac{3(m+p-1)}{2p_0}}} + 1 \right),
\end{aligned} \tag{2.53}$$

where

$$\mu_1 = \frac{\frac{3(m+p-1)}{2p_0} - \frac{3(m+p-1)}{2\frac{q+1}{q}(p+1-m)}}{-\frac{1}{2} + \frac{3(m+p-1)}{2p_0}} = (m+p-1) \frac{\frac{3}{2p_0} - \frac{3}{2\frac{q+1}{q}(p+1-m)}}{-\frac{1}{2} + \frac{3(m+p-1)}{2p_0}} \in (0, 1).$$

On the other hand, with the help of Lemma 2.5 and Lemma 2.4, we conclude that there

exist some positive constants C_3 and C_4 such that

$$\begin{aligned}
\|\nabla c_{\varepsilon}\|_{L^{2(q+1)}(\Omega)}^2 &\leq C_3 \| |\nabla c_{\varepsilon}|^{q-1} D^2 c_{\varepsilon} \|_{L^2(\Omega)}^{\frac{2}{q+1}} \|c_{\varepsilon}\|_{L^{\infty}(\Omega)}^{\frac{2(q-1)}{(2q-1)(q+1)}} + C_3 \|c_{\varepsilon}\|_{L^{\infty}(\Omega)}^2 \\
&\leq C_4 (\| |\nabla c_{\varepsilon}|^{q-1} D^2 c_{\varepsilon} \|_{L^2(\Omega)}^{\frac{2}{q+1}} + 1).
\end{aligned} \tag{2.54}$$

Inserting (2.53)–(2.54) into (2.52) and using the Young inequality and (2.50), we derive that there exist positive constants C_5 and C_6 such that for every $\delta > 0$,

$$\begin{aligned}
J_1 &= C_5 \left(\|n_{\varepsilon}^{\frac{p+m-1}{2}}\|_{L^2(\Omega)}^{\frac{3(\frac{p+1-m}{p_0} - \frac{q}{q+1})}{-\frac{1}{2} + \frac{3(m+p-1)}{2p_0}}} + 1 \right) (\| |\nabla c_{\varepsilon}|^{q-1} D^2 c_{\varepsilon} \|_{L^2(\Omega)}^{\frac{2}{q+1}} + 1) \\
&\leq \delta \int_{\Omega} |n_{\varepsilon}^{\frac{p+m-1}{2}}|^2 + \delta \| |\nabla c_{\varepsilon}|^{q-1} D^2 c_{\varepsilon} \|_{L^2(\Omega)}^2 + C_6 \quad \text{for all } t \in (0, T).
\end{aligned} \tag{2.55}$$

Here we have use the fact that

$$0 < \frac{3[\frac{p+1-m}{p_0} - \frac{1}{q+1}]}{-\frac{1}{2} + \frac{3(m+p-1)}{2p_0}} + \frac{2}{q+1} < 2$$

and

$$\frac{3[\frac{p+1-m}{p_0} - \frac{1}{q+1}]}{-\frac{1}{2} + \frac{3(m+p-1)}{2p_0}} > 0, \frac{2}{q+1} > 0.$$

□

Lemma 2.16. *Assuming that $q > \max\{1, p_0 - 1\}$ If*

$$p > \max\{1, 1 - m + \frac{q+1}{3}, q + 2 - \frac{2p_0}{3} - m\}, \quad (2.56)$$

then for all small $\delta > 0$, we can find a constant $C := C(p, q, p_0, \delta) > 0$ such that

$$\int_{\Omega} n_{\varepsilon}^2 |\nabla c_{\varepsilon}|^{2q-2} + \int_{\Omega} n_{\varepsilon}^2 \leq \delta \int_{\Omega} |n_{\varepsilon}^{\frac{p+m-1}{2}}|^2 + \delta \| |\nabla c_{\varepsilon}|^{q-1} D^2 c_{\varepsilon} \|_{L^2(\Omega)}^2 + C \quad (2.57)$$

for all $t \in (0, T)$, where T is given by (2.49).

Proof. Firstly, in light of Hölder inequality with exponents $\frac{q+1}{q-1}$ and $\frac{q+1}{2}$, we obtain

$$\begin{aligned} J_2 &:= \int_{\Omega} n_{\varepsilon}^2 |\nabla c_{\varepsilon}|^{2q-2} \\ &\leq \left(\int_{\Omega} n_{\varepsilon}^{q+1} \right)^{\frac{2}{q+1}} \left(\int_{\Omega} |\nabla c_{\varepsilon}|^{2(q+1)} \right)^{\frac{1}{q-1}} \\ &= \| n_{\varepsilon}^{\frac{m+p-1}{2}} \|_{L^{\frac{2(q+1)}{m+p-1}}(\Omega)}^{\frac{4}{m+p-1}} \| \nabla c_{\varepsilon} \|_{L^{2(q+1)}(\Omega)}^{(2q-2)}. \end{aligned} \quad (2.58)$$

On the other hand, in view of $p > 1 - m + \frac{q+1}{3}$ and Lemma 2.14 and the Gagliardo–Nirenberg inequality we conclude that there exist positive constants C_1 and C_2 such that

$$\begin{aligned} &\| n_{\varepsilon}^{\frac{m+p-1}{2}} \|_{L^{\frac{2(q+1)}{m+p-1}}(\Omega)}^{\frac{4}{m+p-1}} \\ &\leq C_1 (\| \nabla n_{\varepsilon}^{\frac{p+m-1}{2}} \|_{L^2(\Omega)}^{\mu_3} \| n_{\varepsilon}^{\frac{m+p-1}{2}} \|_{L^{\frac{2p_0}{m+p-1}}(\Omega)}^{(1-\mu_3)} + \| n_{\varepsilon}^{\frac{m+p-1}{2}} \|_{L^{\frac{2p_0}{m+p-1}}(\Omega)})^{\frac{4}{m+p-1}} \\ &\leq C_2 (\| \nabla n_{\varepsilon}^{\frac{p+m-1}{2}} \|_{L^2(\Omega)}^{\frac{4\mu_3}{m+p-1}} + 1). \end{aligned} \quad (2.59)$$

Here

$$\mu_3 = \frac{\frac{3(m+p-1)}{2p_0} - \frac{3(m+p-1)}{2(q+1)}}{-\frac{1}{2} + \frac{3(m+p-1)}{2p_0}} = (m+p-1) \frac{\frac{3}{2p_0} - \frac{3}{2(q+1)}}{-\frac{1}{2} + \frac{3(m+p-1)}{2p_0}} \in (0, 1).$$

On the other hand, according to Lemma 2.5 and the Gagliardo–Nirenberg inequality, we can find some positive constants C_3 and C_4 such that

$$\begin{aligned} \|\nabla c_\varepsilon\|_{L^{2(q+1)}(\Omega)}^{(2q-2)} &\leq C_3 \| |\nabla c_\varepsilon|^{q-1} D^2 c_\varepsilon \|_{L^2(\Omega)}^{\frac{2(q-1)}{q+1}} \|c_\varepsilon\|_{L^\infty(\Omega)}^{\frac{2(q-1)}{q+1}} + C_3 \|c_\varepsilon\|_{L^\infty(\Omega)}^{2q-2} \\ &\leq C_4 (\| |\nabla c_\varepsilon|^{q-1} D^2 c_\varepsilon \|_{L^2(\Omega)}^{\frac{2(q-1)}{q+1}} + 1). \end{aligned} \quad (2.60)$$

Now, observing that

$$0 < \frac{3(\frac{2}{p_0} - \frac{2}{q+1})}{-\frac{1}{2} + \frac{3(m+p-1)}{2p_0}} + \frac{2(q-1)}{q+1} < 2$$

and

$$\frac{3(\frac{2}{p_0} - \frac{2}{q+1})}{-\frac{1}{2} + \frac{3(m+p-1)}{2p_0}} > 0, \quad \frac{2(q-1)}{q+1} > 0,$$

hence, inserting (2.59)–(2.60) into (2.58) and using (2.56) and the Gagliardo–Nirenberg inequality, we derive that for any $\delta > 0$,

$$\begin{aligned} J_2 &\leq C_5 (\| |\nabla n_\varepsilon|^{\frac{p+m-1}{2}} \|_{L^2(\Omega)}^{\frac{3(\frac{2}{p_0} - \frac{2}{q+1})}{-\frac{1}{2} + \frac{3(m+p-1)}{2p_0}} + 1) (\| |\nabla c_\varepsilon|^{q-1} D^2 c_\varepsilon \|_{L^2(\Omega)}^{\frac{2(q-1)}{q+1}} + 1) \\ &\leq \delta \int_\Omega |\nabla n_\varepsilon|^{\frac{p+m-1}{2}}|^2 + \delta \| |\nabla c_\varepsilon|^{q-1} D^2 c_\varepsilon \|_{L^2(\Omega)}^2 + C_6 \quad \text{for all } t \in (0, T). \end{aligned} \quad (2.61)$$

with some positive constants C_5 and C_6 .

Since the integral $\int_\Omega n_\varepsilon^2$ can be estimated similarly upon a straightforward simplification of the above argument, this establishes (2.57). \square

Lemma 2.17. *For any $1 \leq l < \frac{3}{2}$, if*

$$1 < q < \frac{2l+3}{3},$$

then for all small $\delta > 0$, the solution of (2.13) from Lemma 2.2 satisfies

$$\int_\Omega |Du_\varepsilon| |\nabla c_\varepsilon|^{2q} \leq \delta \int_\Omega |Au_\varepsilon|^2 + \delta \|\nabla |\nabla c_\varepsilon|^q\|_{L^2(\Omega)}^2 + C \quad \text{for all } t \in (0, T), \quad (2.62)$$

where a positive constant C depends on p , q and δ .

Proof. Firstly, applying the Hölder inequality leads

$$\begin{aligned} J_3 &:= \int_\Omega |Du_\varepsilon| |\nabla c_\varepsilon|^{2q} \\ &\leq \|Du_\varepsilon\|_{L^{q+1}(\Omega)} \|\nabla c_\varepsilon\|_{L^{2(q+1)}(\Omega)}^{2q}. \end{aligned} \quad (2.63)$$

Due to Lemma 2.7, there exists a positive constant C_1 such that

$$\|Du_\varepsilon\|_{L^l(\Omega)} \leq C_1, \quad (2.64)$$

where $l < \frac{3}{2}$ is the same as Lemma 2.7. Hence, using the Gagliardo–Nirenberg inequality and (2.64), since $q+1 \leq \frac{2l+3}{3} + 1 < 6$ we can find C_1 , C_2 and C_3 such that

$$\begin{aligned} & \|Du_\varepsilon\|_{L^{q+1}(\Omega)} \\ & \leq C_1 \|u_\varepsilon\|_{W^{2,2}(\Omega)}^{\frac{6(q+1-l)}{(q+1)(6-l)}} \|u_\varepsilon\|_{W^{1,l}(\Omega)}^{\frac{(5-q)l}{(q+1)(6-l)}} \\ & \leq C_2 \|Au_\varepsilon\|_{L^2(\Omega)}^{\frac{6(q+1-l)}{(q+1)(6-l)}} \|Du_\varepsilon\|_{L^l(\Omega)}^{\frac{(5-q)l}{(q+1)(6-l)}} \\ & \leq C_3 \|Au_\varepsilon\|_{L^2(\Omega)}^{\frac{6(q+1-l)}{(q+1)(6-l)}}. \end{aligned} \quad (2.65)$$

Now, in light of Lemma 2.5 and the Gagliardo–Nirenberg inequality that we have

$$\begin{aligned} \|\nabla c_\varepsilon\|_{L^{2(q+1)}(\Omega)}^{2q} & \leq C_4 (\|\nabla c_\varepsilon\|_{L^2(\Omega)}^{q-1} \|D^2 c_\varepsilon\|_{L^2(\Omega)}^{\frac{2q[2(2q+2)-6]}{(2q-1)(2q+2)}} \|c_\varepsilon\|_{L^\infty(\Omega)}^{\frac{2q[6q-2(q+1)]}{(2q-1)(2q+2)}} + \|c_\varepsilon\|_{L^\infty(\Omega)}^{2q}) \\ & \leq C_5 (\|\nabla c_\varepsilon\|_{L^2(\Omega)}^{q-1} \|D^2 c_\varepsilon\|_{L^2(\Omega)}^{\frac{2q}{q+1}} + 1), \end{aligned} \quad (2.66)$$

where C_4 and C_5 are positive constants independent of ε . Inserting (2.59)–(2.60) into (2.58) and using $q < \frac{2l+3}{3}$ and the Young inequality, we have

$$\begin{aligned} J_3 & \leq C_6 \|Au_\varepsilon\|_{L^2(\Omega)}^{\frac{6(q+1-l)}{(q+1)(6-l)}} (\|\nabla c_\varepsilon\|_{L^2(\Omega)}^{q-1} \|D^2 c_\varepsilon\|_{L^2(\Omega)}^{\frac{2q}{q+1}} + 1) \\ & \leq \delta \int_\Omega |Au_\varepsilon|^2 + \delta \|\nabla c_\varepsilon\|_{L^2(\Omega)}^{q-1} \|D^2 c_\varepsilon\|_{L^2(\Omega)}^2 + C_7 \quad \text{for all } t \in (0, T). \end{aligned} \quad (2.67)$$

with some positive constants C_6 and C_7 . □

Lemma 2.18. *Assuming that $q > 1$. If*

$$\|Du_\varepsilon(\cdot, t)\|_{L^2(\Omega)} \leq K \quad \text{for all } t \in (0, T), \quad (2.68)$$

then for all small $\delta > 0$, the solution of (2.13) from Lemma 2.2 satisfies

$$\int_\Omega |Du_\varepsilon| |\nabla c_\varepsilon|^{2q} \leq \delta \|\nabla |\nabla c_\varepsilon|^q\|_{L^2(\Omega)}^2 + C \quad \text{for all } t \in (0, T), \quad (2.69)$$

where a positive constant C depends on q and δ .

Proof. Firstly, using the Hölder inequality, we find

$$\begin{aligned} J_3 & := \int_\Omega |Du_\varepsilon| |\nabla c_\varepsilon|^{2q} \\ & \leq \|Du_\varepsilon\|_{L^2(\Omega)} \|\nabla c_\varepsilon\|_{L^{4q}(\Omega)}^{2q} \\ & \leq K \|\nabla c_\varepsilon\|_{L^{4q}(\Omega)}^{2q} \quad \text{for all } t \in (0, T). \end{aligned} \quad (2.70)$$

Now, it then follows from Lemma 2.5 and the Gagliardo–Nirenberg inequality that we have

$$\begin{aligned}\|\nabla c_\varepsilon\|_{L^{4q}(\Omega)}^{2q} &\leq C_1(\|\nabla c_\varepsilon\|^{q-1} D^2 c_\varepsilon\|_{L^2(\Omega)}^{\frac{4q[4q-3]}{(2q-1)4q}} \|c_\varepsilon\|_{L^\infty(\Omega)}^{\frac{2q[6q-4q]}{(2q-1)4q}} + \|c_\varepsilon\|_{L^\infty(\Omega)}^{2q}) \\ &\leq C_2(\|\nabla c_\varepsilon\|^{q-1} D^2 c_\varepsilon\|_{L^2(\Omega)}^{\frac{2q-3}{2q-1}} + 1),\end{aligned}\tag{2.71}$$

where some positive constants C_1, C_2 . Inserting (2.71) into (2.70) and employing the Young inequality, we can get (2.69). \square

Lemma 2.19. *Assuming that $m > \frac{9}{8}$. Then for all $p > 1$ and $q > 1$,*

$$\|n_\varepsilon(\cdot, t)\|_{L^p(\Omega)} + \|\nabla c_\varepsilon(\cdot, t)\|_{L^{2q}(\Omega)} \leq C \text{ for all } t \in (0, T),\tag{2.72}$$

where T is given by (2.49).

Proof. We only need to prove case $2 \geq m > \frac{9}{8}$. Since $m > 2$, employing almost exactly the same arguments as in the proof of Lemma 4.1 in [27] (see also [22]) (the minor necessary changes are left as an easy exercise to the reader), we conclude the estimate (2.72).

Case $2 \geq m > \frac{9}{8}$. We divide the proof into two steps.

Step 1. We first make sure that there exists $C_1 > 0$ such that

$$\|Du_\varepsilon(\cdot, t)\|_{L^2(\Omega)} \leq C_1 \text{ for all } t \in (0, T).\tag{2.73}$$

To achieve this, in light of $m > \frac{9}{8}$, $p_0 \geq 1$ and $q > 1$, then

$$\begin{aligned}[2(m-1) + \frac{2p_0}{3}]q + m - 1 &> 3(m-1) + \frac{2p_0}{3} \\ &> 3 \times \frac{1}{9} + \frac{2}{3} = 1,\end{aligned}\tag{2.74}$$

$$\begin{aligned}[2(m-1) + \frac{2p_0}{3}]q + m - 1 &> m - 1 + \frac{2p_0}{3}q \\ &> m - 1 + p_0 \frac{q}{q+1}\end{aligned}\tag{2.75}$$

and

$$\begin{aligned}\{[2(m-1) + \frac{2p_0}{3}]q + m - 1\} - \{\frac{q+1}{3} + 1 - m\} &= 2(m-1)(q-1) + (\frac{2p_0}{3} - \frac{1}{3})q \\ &> \frac{1}{3}q > 0.\end{aligned}\tag{2.76}$$

Now, since $p_0 \in [1, 9(m-1))$ is arbitrary, with the help of $m > \frac{9}{8}$, we can finally pick $p_0 < 9(m-1)$ sufficiently close to $9(m-1)$ such that

$$2(m-1) + \frac{2p_0}{3} - 1 > 0,$$

which implies that for all $q > 1$ and $p_0 < 9(m-1)$ sufficiently close $9(m-1)$ to

$$q + 2 - \frac{2p_0}{3} - m < [2(m-1) + \frac{2p_0}{3}]q + m - 1, \quad (2.77)$$

which along with (2.74)–(2.76) yields that

$$\begin{aligned} & \max\{1, m-1 + p_0 \frac{q}{q+1}, \frac{q+1}{3} + 1 - m, q + 2 - \frac{2p_0}{3} - m\} \\ & < [2(m-1) + \frac{2p_0}{3}]q + m - 1 \quad \text{for all } q > 1. \end{aligned} \quad (2.78)$$

Now, in view of $l < \frac{3}{2}$ is arbitrary, we can finally pick $q_0 < 2$ sufficiently close to 2 such that $q_0 < \frac{2l+3}{3}$. Now, we choose $q = q_0$ in (2.78),

$$p \in (\max\{1, m-1 + p_0 \frac{q_0}{q_0+1}, \frac{q_0+1}{3} + 1 - m, q_0 + 2 - \frac{2p_0}{3} - m\}, [2(m-1) + \frac{2p_0}{3}]q_0 + m - 1)$$

and δ small enough in Lemma 2.13–Lemma 2.17, then by Lemma 2.11, we conclude that

$$\begin{aligned} & \frac{d}{dt} \left(\int_{\Omega} n_{\varepsilon}^p + \int_{\Omega} |\nabla c_{\varepsilon}|^{2q_0} + \int_{\Omega} |A^{\frac{1}{2}} u_{\varepsilon}|^2 \right) + \frac{(p-1)}{4C_D} \int_{\Omega} n_{\varepsilon}^{m+p-3} |\nabla n_{\varepsilon}|^2 \\ & + \frac{(q_0-1)}{q_0^2} \int_{\Omega} |\nabla |\nabla c_{\varepsilon}|^{q_0}|^2 + \frac{1}{4} \int_{\Omega} |\nabla c_{\varepsilon}|^{2q_0-2} |D^2 c_{\varepsilon}|^2 + \int_{\Omega} |A u_{\varepsilon}|^2 \\ & \leq C_2 \end{aligned} \quad (2.79)$$

with some positive constant C_2 . Assuming that $y := \int_{\Omega} n_{\varepsilon}^p + \int_{\Omega} |\nabla c_{\varepsilon}|^{2q_0} + \int_{\Omega} |A^{\frac{1}{2}} u_{\varepsilon}|^2$, then (2.79) implies that there exist positive constants C_3 and C_4 such that

$$\frac{d}{dt} y(t) + C_3 y^h(t) \leq C_4,$$

where h is a positive constant. Then an ODE comparison argument yields that there exists a positive constant C_5 independent of ε such that

$$\|n_{\varepsilon}(\cdot, t)\|_{L^p(\Omega)} + \|\nabla c_{\varepsilon}(\cdot, t)\|_{L^{2q_0}(\Omega)} + \|A^{\frac{1}{2}} u_{\varepsilon}(\cdot, t)\|_{L^2(\Omega)} \leq C_5 \quad \text{for all } t \in (0, T), \quad (2.80)$$

which together with $q_0 < 2$, $p < [2(m-1) + \frac{2p_0}{3}]q_0 + m - 1$, $p_0 < 9(m-1)$ sufficiently close to $9(m-1)$ and $2 \geq m > \frac{9}{8}$ imply that

$$\|n_{\varepsilon}(\cdot, t)\|_{L^{\frac{7}{4}}(\Omega)} \leq C_6 \quad \text{for all } t \in (0, T). \quad (2.81)$$

with some positive constant C_6 . Therefore, by Lemma 2.1, we can get (2.73).

Step 2. We proceed to prove the statement of the lemma. To this end, by (2.73), we can get (2.68), hence, (2.69) holds. Now, inserting (2.51), (2.57), (2.62) and (2.69) into (2.45) and using (2.78) yields to

$$\begin{aligned} & \frac{d}{dt} \left(\int_{\Omega} n_{\varepsilon}^p + \int_{\Omega} |\nabla c_{\varepsilon}|^{2q} \right) + \frac{(p-1)}{4C_D} \int_{\Omega} n_{\varepsilon}^{m+p-3} |\nabla n_{\varepsilon}|^2 \\ & + \frac{(q-1)}{q^2} \int_{\Omega} |\nabla |\nabla c_{\varepsilon}|^q|^2 + \frac{1}{4} \int_{\Omega} |\nabla c_{\varepsilon}|^{2q-2} |D^2 c_{\varepsilon}|^2 \\ & \leq C_7, \end{aligned} \quad (2.82)$$

where C_7 is a positive constant independent of ε . Here we have picked the δ small enough. Employing the same arguments as in the proof of step 1, we conclude (2.72). The proof of Lemma 2.19 is complete. \square

3 The proof of main results

In preparation of an Aubin-Lions type compactness argument, we intend to supplement Lemma 2.19 with bounds on time-derivatives. With these estimates, we can construct weak solutions by means of a standard extraction procedure, therefore, we only give general idea, one can see [27, 22] for more details.

The proof of Theorem 2.1 Firstly, choosing $p = 2$ in (2.72), we derive that

$$\|n_{\varepsilon}(\cdot, t)\|_{L^2(\Omega)} \leq C_1 \quad \text{for all } t \in (0, T) \quad (3.1)$$

with some positive constant C_1 . Now, in view of (3.1), employing the variation-of-constants formula for u_{ε} and by the properties of the Stokes semigroup, we conclude that there exists a positive constant C_2 such that

$$\|A^{\gamma} u_{\varepsilon}(\cdot, t)\|_{L^2(\Omega)} \leq C_2 \quad \text{for all } t \in (0, T), \quad (3.2)$$

where $\gamma \in (\frac{3}{4}, 1)$ and T are the same as (2.7) and (2.49), respectively. Since, $D(A^{\gamma})$ is continuously embedded into $L^{\infty}(\Omega)$, hence, in view of (3.2), we may find that

$$\|u_{\varepsilon}(\cdot, t)\|_{L^{\infty}(\Omega)} \leq C_3 \quad \text{for all } t \in (0, T). \quad (3.3)$$

for some positive constant C_3 . Next, choosing $q = 2$ in (2.72), we get that

$$\|\nabla c_\varepsilon(\cdot, t)\|_{L^4(\Omega)} \leq C_4 \quad \text{for all } t \in (0, T), \quad (3.4)$$

where C_4 is positive constant independent of ε . Hence, in light of (3.3)–(3.4) and the L^p - L^q estimates associated heat semigroup, we derive that there exists a positive constant C_5 such that

$$\|c_\varepsilon(\cdot, t)\|_{W^{1,\infty}(\Omega)} \leq C_5 \quad \text{for all } t \in (0, T). \quad (3.5)$$

Now, due to (3.3) and (3.5), we may use the standard Moser-type iteration to conclude that there exists a positive constant C_6 such that

$$\|n_\varepsilon(\cdot, t)\|_{L^\infty(\Omega)} \leq C_6 \quad \text{for all } t \in (0, T). \quad (3.6)$$

Next, with the help of (3.3), (3.5) and (3.6), for all $\varepsilon \in (0, 1)$, we can fix a positive constants C_7 such that

$$n_\varepsilon \leq C_7, |\nabla c_\varepsilon| \leq C_7 \quad \text{and} \quad |u_\varepsilon| \leq C_7 \quad \text{in } \Omega \times (0, T). \quad (3.7)$$

Now, employing the same argument of Lemma 5.1 of [27] (see also Lemmata 3.22 and 3.23 of [22]), we may derive that there exists a positive constant C_8 such that

$$\|\partial_t n_\varepsilon(\cdot, t)\|_{(W_0^{2,2}(\Omega))^*} \leq C_8 \quad \text{for all } t > 0 \quad (3.8)$$

as well as

$$\int_0^T \|\partial_t n_\varepsilon^\varsigma(\cdot, t)\|_{(W_0^{3,2}(\Omega))^*} dt \leq C_8(T+1) \quad (3.9)$$

and

$$\int_0^T \int_\Omega n_\varepsilon^{m+p-3} |\nabla n_\varepsilon|^2 \leq C_8 T, \quad (3.10)$$

where $p := 2\zeta - m + 1$ and $\varsigma > m$ satisfying $\varsigma \geq 2(m-1)$.

Now, in conjunction with (3.2), (3.3), (3.5) and (3.6) and the Aubin–Lions compactness lemma ([11]), we thus infer that there exists a sequence of numbers $\varepsilon := \varepsilon_j \searrow 0$ such that

$$n_\varepsilon \rightharpoonup n \quad \text{weakly star in } L_{loc}^\infty(\Omega \times (0, \infty)), \quad (3.11)$$

$$n_\varepsilon \rightarrow n \quad \text{in } L_{loc}^\infty([0, \infty); (W_0^{3,2}(\Omega))^*), \quad (3.12)$$

$$c_\varepsilon \rightarrow c \quad \text{a.e. in } \Omega \times (0, \infty), \quad (3.13)$$

$$\nabla c_\varepsilon \rightarrow \nabla c \quad \text{in } L_{loc}^\infty(\bar{\Omega} \times [0, \infty)), \quad (3.14)$$

$$u_\varepsilon \rightarrow u \quad \text{a.e. in } \Omega \times (0, \infty), \quad (3.15)$$

and

$$Du_\varepsilon \rightharpoonup Du \quad \text{weakly in } L_{loc}^\infty(\Omega \times [0, \infty)) \quad (3.16)$$

holds for some triple (n, c, u) .

Since $p = 2\zeta - m + 1$, then by (3.10) implies that for each $T > 0$, $(n_\varepsilon^\zeta)_{\varepsilon \in (0,1)}$ is bounded in $L^2((0, T); W^{1,2}(\Omega))$. With the help of (3.9), we also show that

$$(\partial_t n_\varepsilon^\zeta)_{\varepsilon \in (0,1)} \text{ is bounded in } L^1((0, T); (W_0^{3,2}(\Omega))^*) \text{ for each } T > 0.$$

Hence, an Aubin-Lions lemma (see e.g. [11]) applies to the above inequality we have the strong precompactness of $(n_\varepsilon^\zeta)_{\varepsilon \in (0,1)}$ in $L^2(\Omega \times (0, T))$. Therefore, we can pick a suitable subsequence such that $n_\varepsilon^\zeta \rightarrow z^\zeta$ for some nonnegative measurable $z : \Omega \times (0, \infty) \rightarrow \mathbb{R}$. In light of (3.11) and the Egorov theorem, we have $z = n$ necessarily, so that

$$n_\varepsilon \rightarrow n \quad \text{a.e. in } \Omega \times (0, \infty) \quad (3.17)$$

is valid. Now, in light of (3.11), (3.14), and (3.16), we derive that (2.9) is hold.

Now, we will prove that (n, c, u) is the global weak solution of (1.1). To this end, by (3.17), (3.11), (3.13), (3.14) and (3.15), we have n, c are nonnegative and (2.1) and (2.2) are valid. Next, with the help of the standard arguments, letting $\varepsilon = \varepsilon_j \searrow 0$ in the approximate system (2.13) and using (3.17)–(3.11) and (3.13)–(3.16), we can get (2.3)–(2.5).

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